Symmetric Lorentz-Minkowski, Antisymmetric Dirac-Majorana

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Abstract

We show a nice symmetric/antisymmetric relation between the four vector Lorentz transformation and the Dirac spinor one in the Majorana representation. From the spinor one, we exhibit the antisymmetric pending of the symmetric Minkowski metric. We then rewrite the Dirac equation in various ways exploiting group properties induced by these relations, and this without complex numbers. We show also a nice relation with a five dimensional metric. When done, we will see that the traditional complex electromagnetic coupling could be handled also without complex numbers by just considering two coupled real fields instead of one complex field. Finally, we will show that going toward six or ten dimensional spacetime would be more natural from a group point of view.

1 The coordinate Lorentz transformation, the symmetric Minkowski matrix

A coordinate Lorentz transformation on a tuple $x^{\mu=0,1,2,3}$ of four real numbers can be written:

$$\mathscr{L}^{c}(A|x)^{\mu} \stackrel{\text{def}}{=} (e^{A\eta})^{\mu}_{\nu} x^{\nu} \tag{1}$$

with the Minkowski η 4x4 real symmetric matrix being:

$$\eta \stackrel{\text{def}}{=} \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right)$$

and A being a 4x4 real antisymmetric matrix. Concerning the exponential, someone must note the pattern: antisymmetric matrix A, parametrizing a Lorentz transformation, multiplied by the constant η 4x4 real symmetric matrix. A 4x4 real antisymmetric matrix having six free parameters, we recover the number of parameters (three rotations plus three boosts) of a Lorentz transformation.

2 Scalar, vector Lorentz transformations

On a field $\phi(x)$ (no index), a tuple $V^{\mu=0,1,2,3}(x)$ of fields, they are defined with:

$$\mathscr{L}^{s}(A|\phi)(\mathscr{L}^{c}(A|x)) \stackrel{\text{def}}{=} \phi(x)$$
$$\mathscr{L}^{\bullet}(A|V)^{\mu}(\mathscr{L}^{c}(A|x)) \stackrel{\text{def}}{=} (e^{A\eta})^{\mu}_{\nu}V^{\nu}(x)$$
(2)

We can also introduce the "down" vector transformation on a $V_{\mu=0,1,2,3}(x)$:

$$\mathscr{L}_{\bullet}(A|V)_{\mu}(\mathscr{L}^{c}(A|x)) \stackrel{\text{def}}{=} (e^{-A\eta})^{\nu}_{\mu}V_{\nu}(x)$$

3 η as a metric

If the length of a $V^{\mu}(x)$ is defined, in matrix notation, with:

$$l[\eta](V)(x) \stackrel{\text{\tiny def}}{=} {}^{\mathrm{t}}V(x)\eta V(x) \stackrel{\text{\tiny def}}{=} ({}^{\mathrm{t}}V(x))_{\mu}\eta^{\mu}_{\nu}V^{\nu}(x)$$

(t for the transposition operation), it can be shown easily that it is a Lorentz invariant:

$$l[\eta](\mathscr{L}^{\bullet}(A|V))(\mathscr{L}^{c}(A|x)) \stackrel{\text{dem}}{=} l[\eta](V)(x)$$

due to the fact that:

$${}^{t}(e^{A\eta})\eta e^{A\eta} \stackrel{\text{dem}}{=} \eta$$

It is a general property that if A is an antisymmetric square matrix and S a symmetric square matrix of same dimension, we have:

$${}^{t}(e^{AS})Se^{AS} \stackrel{\text{dem}}{=} e^{-SA}e^{SA}S \stackrel{\text{dem}}{=} S$$
$${}^{t}(e^{SA})Ae^{SA} \stackrel{\text{dem}}{=} e^{-AS}e^{AS}A \stackrel{\text{dem}}{=} A$$

This second property will have some importance in the following. If S and A are invertible, we have also:

$$e^{AS}S^{-1}{}^{t}(e^{AS}) \stackrel{\text{dem}}{=} e^{AS}S^{-1}e^{-SA} \stackrel{\text{dem}}{=} e^{AS}e^{-AS}S^{-1} \stackrel{\text{dem}}{=} S^{-1}$$
(3)
$$e^{SA}A^{-1}{}^{t}(e^{SA}) \stackrel{\text{dem}}{=} e^{SA}A^{-1}e^{-AS} \stackrel{\text{dem}}{=} e^{SA}e^{-SA}A^{-1} \stackrel{\text{dem}}{=} A^{-1}$$

4 Dirac spinor Lorentz transformation

On a tuple $\psi^{\alpha=0,1,2,3}(x)$ of fields, we define the "spinor" Lorentz transformation with:

$$\mathscr{L}^{\circ}(A|\psi)^{\alpha}(\mathscr{L}^{c}(A|x)) \stackrel{\text{def}}{=} (e^{\Gamma[A]})^{\alpha}_{\beta}\psi^{\beta}(x)$$
(4)

with $\Gamma[A]$ being:

$$\Gamma[A] \stackrel{\text{\tiny def}}{=} \frac{1}{8} [\gamma^{\mu}, \gamma^{\nu}] (\eta A \eta)^{\mu}_{\nu}$$

with the $\gamma^{\mu=0,1,2,3}$ being four 4x4 complex matrices verifying:

$$\{\gamma^{\mu}, \gamma^{\nu}\} = 2(\eta^{-1})^{\mu}_{\nu}I \tag{5}$$

with I being the four dimensional identity matrix.

Presented in this form, the $e^{\Gamma[A]}$ appearing in formula (4) is less appealing than the $e^{A\eta}$ appearing in (2); it looks a little bit exotic and much more complicated, but we are going to show that in fact it can be presented with the same similar quite simpler structure than in $e^{A\eta}$.

About the definition and properties of $\Gamma[A]$, have a look to Appendix B with $S = \eta$. In particular we have the two important properties:

$$e^{\Gamma[A]}\gamma^{\mu}e^{-\Gamma[A]} \stackrel{\text{dem}}{=} (e^{-A\eta})^{\mu}_{\nu}\gamma^{\nu}$$
(6)

$$e^{\Gamma[A]}\gamma_{\mu}e^{-\Gamma[A]} \stackrel{\text{dem}}{=} (e^{A\eta})^{\nu}_{\mu}\gamma_{\nu} \tag{7}$$

with the down γ_{μ} being:

$$\gamma_{\mu} \stackrel{\text{def}}{=} \eta^{\nu}_{\mu} \gamma^{\nu}$$

Since the $e^{\Gamma[A]}$ matrix looks complex, up so far, the $\psi^{\alpha}(x)$ have to be complex numbers too.

5 The Dirac equation

Over a $\psi^{\alpha}(x)$ tuple, it reads:

$$i(\gamma^{\mu})^{\alpha}_{\beta}\partial_{\mu}\psi^{\beta}(x) = \frac{mc}{\hbar}\psi^{\alpha}(x)$$

or in a more compact matrix notation:

$$i\gamma^{\mu}\partial_{\mu}\psi(x) = \frac{mc}{\hbar}\psi(x) \tag{8}$$

By using (1), (4) and exploiting (6) we can show quite easily that it is a Lorentz invariant.

6 Terminology, notation

In this article, the words "coordinate", "scalar", "vector", "tensor", "spinor", etc are used as a qualifier for a transformation and not to define a tuple/matrix of numbers or functions/fields as for x^{μ} , $\phi(x)$, $V^{\mu}(x)$, $\psi^{\mu}(x)$ or later $g_{\mu\nu}(x)$.

On an advanced theory, the same tuple/matrix of numbers/functions may be subjected to various transformations, and then labeling them with such qualifiers can lead to confusion. (For example, we can see in Appendix D that the tuple of functions $\psi^{\alpha}(x)$ is subjected to a scalar transformation for \mathscr{R} but to a spinor one for the \mathscr{L}_{I} transformation).

It is only in a context where only one transformation is around that we can displace such qualifer toward a tuple/matrix, and then speak for example of the $A_{\mu}(x)$ "vector" electromagnetic potential or the $\psi(x)$ "spinor". (In this text, we stick to the tradition of noting a tuple subjected to a vector Lorentz transformation \mathscr{L}^{\bullet} with an uppercase latin letter, as V(x), and of using the lowercase greek letter $\psi(x)$ for a tuple submitted to a spinor Lorentz transformation \mathscr{L}°).

For the transformations, we use a notation that may look heavy, but which is in general complete in the sense that it carries all the needed informations:

$$\mathscr{T}^{qualifier}(parameters|whatever)^{indices}(arguments)$$

with "qualifier" that could be upward or downward, "parameters" being a set of tuples/matrices of numbers/functions and "whatever" being a tuple/matrix of numbers or functions submitted to the transformation. The result of a transformation being also a tuple/matrix of numbers/functions, it has in general upward/downward indices and ending brackets with arguments.

For functions/fields depending of parameters, we use:

name[parameters](arguments)

We use also $\stackrel{\text{def}}{=}$ in case an equality is a definition (left side is defined by the right side), and $\stackrel{\text{dem}}{=}$ when an equality comes from a demonstration (left side is demonstrated to be the right side). We have also $\stackrel{\text{cas}}{=}$ in case an equality is demonstrated by using a computer algebra system (CAS).

We definitely avoid the practice of setting $\hbar = c = 1$ which complicates the reading of the dimensionality of quantities.

Let us go now to the core of this article.

7 The A_a basis, Γ_a matrices and θ^a parametrization

Let us define the antisymmetric $A_{a=1,2,3,4,5,6}$ six real matrices with:

These matrices form a basis for any 4x4 real antisymmetric matrix (see Appendix *B* for some of their properties).

Instead of parametrizing a Lorentz transformation with a A antisymmetric matrix, we can do it by using six real numbers θ^a so that:

$$A = \theta^a A_a \Leftrightarrow \theta^a \stackrel{\text{dem}}{=} -\frac{1}{2} Tr(A_a A)$$

With this choice for the A_a matrices, the tuple $(\theta^{1,2,3}, 0, 0, 0)$ parametrizes a spatial rotation, whilst $(0,0,0,\theta^{4,5,6})$ parametrizes a boost.

By using also the Γ_a matrices defined with:

$$\Gamma_a \stackrel{\text{def}}{=} \frac{1}{8} [\gamma_\mu, \gamma_\nu] (A_a)^{\mu}_{\nu}$$

(see also Appendix B with $S = \eta$), we can write now:

$$\mathscr{L}^{c}(\theta|x)^{\mu} \stackrel{\text{def}}{=} (e^{\theta^{a}A_{a}\eta})^{\mu}_{\nu}x^{\nu}$$
$$\mathscr{L}^{\bullet}(\theta|V)^{\mu}(\mathscr{L}^{c}(\theta|x)) \stackrel{\text{def}}{=} (e^{\theta^{a}A_{a}\eta})^{\mu}_{\nu}V^{\nu}(x)$$
$$\mathscr{L}^{\circ}(\theta|\psi)^{\alpha}(\mathscr{L}^{c}(\theta|x)) \stackrel{\text{def}}{=} (e^{\theta^{a}\Gamma_{a}})^{\alpha}_{\beta}\psi^{\beta}(x)$$

8 Hidden Dirac-Majorana antisymmetric ξ matrix

In the Majorana representation, the $\gamma^{\mu=0,1,2,3}$ matrices are (see Appendix A):

$$\gamma^{0} \stackrel{\text{def}}{=} i \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \gamma^{1} \stackrel{\text{def}}{=} i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\gamma^{2} \stackrel{\text{def}}{=} i \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \gamma^{3} \stackrel{\text{def}}{=} i \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

If now defining the real antisymmetric matrix:

$$\xi \stackrel{\text{def}}{=} \left(\begin{array}{ccc} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array} \right) \stackrel{\text{dem}}{=} i\gamma^0$$

we can show the remarkable fact that:

$$\Gamma_a \stackrel{\text{\tiny dem}}{=} S_a \xi$$

with:

$$S_{1} \stackrel{\text{def}}{=} \frac{1}{2} \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad S_{2} \stackrel{\text{def}}{=} \frac{1}{2} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$
$$S_{3} \stackrel{\text{def}}{=} \frac{1}{2} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad S_{4} \stackrel{\text{def}}{=} \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$
$$S_{5} \stackrel{\text{def}}{=} \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad S_{6} \stackrel{\text{def}}{=} \frac{1}{2} \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

being all real symmetric matrices!

We have the remarkable fact that the coordinate, vector and spinor Lorentz transformations with the θ^a as parameters, now look like:

$$\mathscr{L}^{c}(\theta|x)^{\mu} \stackrel{\text{def}}{=} (e^{\theta^{a}A_{a}\eta})^{\mu}_{\nu}x^{\nu}$$
$$\mathscr{L}^{\bullet}(\theta|V)^{\mu}(\mathscr{L}^{c}(\theta|x)) \stackrel{\text{def}}{=} (e^{\theta^{a}A_{a}\eta})^{\mu}_{\nu}V^{\nu}(x)$$
(9)

$$\mathscr{L}^{\circ}(\theta|\psi)^{\alpha}(\mathscr{L}^{c}(\theta|x)) \stackrel{\text{def}}{=} (e^{\theta^{a}S_{a}\xi})^{\alpha}_{\beta}\psi^{\beta}(x)$$
(10)

We note the remarkable and nice interchange of symmetric/antisymmetric real matrices between (9) and (10) with:

$$(A_a, \eta) \Leftrightarrow (S_a, \xi)$$
 $(antisyms, sym) \Leftrightarrow (syms, antisym)$

To spot this interchange, we can write the vector/spinor group commutators together:

$$\begin{bmatrix} A_a \eta, A_b \eta \end{bmatrix} \stackrel{\text{dem}}{=} l_{ab}{}^c A_c \eta \tag{11}$$
$$\begin{bmatrix} S_a \xi, S_b \xi \end{bmatrix} \stackrel{\text{dem}}{=} l_{ab}{}^c S_c \xi$$

with the same $l_{ab}{}^c$ real constants (see Appendices *B* and *C*).

We note also that since $e^{\theta^a S_a \xi}$ is real, the $\psi^{\alpha}(x)$ tuple does not have to be complex and can stay a priori real.

9 Four symmetric matrices are lacking

The set S_a , despite of being linearly independent, is not a complete set to form a basis of the set of symmetric 4x4 real matrices $\mathscr{S}(4,\mathbb{R})$. We should have ten matrices instead of six; four are lacking! Four? but we have four γ_{μ} , and if ... yes! It can be shown that:

$$\gamma_{\mu} \stackrel{\text{dem}}{=} -i\tilde{S}_{\mu}\xi \tag{12}$$

with:

$$\tilde{S}_{0} \stackrel{\text{def}}{=} I \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \tilde{S}_{1} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$
$$\tilde{S}_{2} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \tilde{S}_{3} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

being all symmetric too!

We note that (7) can be written:

$$e^{\theta^a S_a \xi} \left(\tilde{S}_\mu \xi \right) e^{-\theta^b S_b \xi} \stackrel{\text{dem}}{=} \left(e^{\theta^c A_c \eta} \right)^{\nu}_{\mu} \left(\tilde{S}_\nu \xi \right)$$
(13)

We note also that we have:

$$\{\tilde{S}_{\mu}\xi,\tilde{S}_{\nu}\xi\}\stackrel{\text{\tiny dem}}{=}-2\eta^{\mu}_{\nu}I$$

10 The $\tilde{S}_{\alpha}\xi$ Dirac equation

We can rewrite now the Dirac equation as:

$$(\eta^{-1})^{\mu}_{\alpha}\tilde{S}_{\alpha}\xi\,\partial_{\mu}\psi(x) = \frac{mc}{\hbar}\psi(x) \tag{14}$$

Since all matrices are real, the $\psi^{\alpha}(x)$ tuple is a priori real.

For reasons that will appear clear later, we are going to rewrite it as:

$$h^{\mu\alpha}(x)\,\tilde{S}_{\alpha}\xi\,\partial_{\mu}\psi(x) = \frac{mc}{\hbar}\psi(x) \tag{15}$$

with the $h^{\mu\alpha}(x)$ real fields being constant and defined as:

$$h^{\mu\alpha}(x) \stackrel{\text{\tiny def}}{=} h^{\mu\alpha} \stackrel{\text{\tiny def}}{=} (\eta^{-1})^{\mu}_{\alpha} \stackrel{\text{\tiny dem}}{=} \eta^{\mu}_{\alpha}$$

If the $h^{\mu\alpha}(x)$ fields are Lorentz transformed as:

$$\mathscr{L}^{\bullet\bullet}(\theta|h)^{\mu\alpha}(\mathscr{L}^{c}(\theta|x)) \stackrel{\text{\tiny def}}{=} (e^{\theta^{a}A_{a}\eta})^{\mu}_{\nu}(e^{\theta^{b}A_{b}\eta})^{\alpha}_{\beta}h^{\nu\beta}(x)$$

we have, due to (3):

$$\mathscr{L}^{\bullet\bullet}(\theta|h)^{\mu\alpha}(x) \stackrel{\mathrm{dem}}{=} (\eta^{-1})^{\mu}_{\alpha}$$

and we can show, with the help of (1), (10), (13), that (15) is Lorentz invariant. The introduction of the h(x) fields may look cumbersome, but we do this to be able to compare with other Dirac like equations coming in further sections.

11 The $S_a\xi$ more natural Dirac like equation

Indeed, from the group point of view, since:

$$[S_a\xi, S_b\xi] \stackrel{\text{\tiny dem}}{=} l_{ab}{}^c S_c\xi$$

and by defining the L_a "adjoint" six 6x6 real matrices:

$$(L_a)_c^b \stackrel{\text{\tiny def}}{=} l_{ac}^{\ b}$$

we have the exponential relation:

$$e^{\theta^c S_c \xi} S_a \xi \, e^{-\theta^d S_d \xi} \stackrel{\text{dem}}{=} (e^{\theta^e L_e})^b_a S_b \xi$$

So it would be much more natural to introduce the equation:

$$h^{\mu a}(x) S_a \xi \,\partial_\mu \psi(x) = \frac{mc}{\hbar} \psi(x) \tag{16}$$
$$\equiv h^{\mu a}(x) \Gamma_a \partial_\mu \psi(x) = \frac{mc}{\hbar} \psi(x)$$

This equation would be Lorentz invariant if the, a priori real, $h^{\mu a}(x)$ fields be Lorentz transformed as:

$$\mathscr{L}^{\bullet\square}(\theta|h)^{\mu a}(\mathscr{L}^{c}(\theta|x)) \stackrel{\text{def}}{=} (e^{\theta^{c}A_{c}\eta})^{\mu}_{\nu}(e^{\theta^{d}L_{d}})^{a}_{b}h^{\nu b}(x)$$
(17)

It is a remarkable trick from Dirac that the lonely relation (5), leading to (6) avoided to consider the more natural equation (16). Once having written (6), the Dirac equation (8), crowded with complex numbers, looks an ad hoc construction from the group point of view. Someone may argue that (16) introduces the extra fields $h^{\mu a}(x)$ that may look cumbersome, but it is not new to see the introduction of coworking real fields related to the Dirac equation; this had been introduced already, through "vierbein", by people wanting to handle the Dirac equation within the framework of general relativity (see Appendix D). We see here that coworking fields emerged naturally lonely from a group point of view.

12 The $A_a\eta$ Dirac like equation

At this point, it is worth noting that we could introduce also the equation:

$$h^{\mu a}(x)(A_a\eta)^{\nu}_{\lambda}\partial_{\mu}V^{\lambda}(x) = \frac{mc}{\hbar}V^{\nu}(x)$$
$$h^{\mu a}(x)A_a\eta\,\partial_{\mu}V(x) = \frac{mc}{\hbar}V(x)$$
(18)

which would be also Lorentz invariant with the same (17) transformation for the h(x) fields, the (2) one for the $V^{\mu}(x)$ fields, and by exploiting:

$$e^{\theta^c A_c \eta} A_a \eta \, e^{-\theta^d A_d \eta} \stackrel{\text{dem}}{=} (e^{\theta^e L_e})^b_a A_b \eta$$

induced by (11). We are not aware of any attempt to do physics with such version of the Dirac equation which is invariant by vector Lorentz transformation only.

13 The ten Σ_A matrices

We can show that the set of ten matrices Σ_A such that:

$$\Sigma_{A=1,10} \stackrel{\text{def}}{=} \frac{1}{2} \tilde{S}_{0,1,2,3}, S_{a=1,2,3,4,5,6}$$

forms a basis of $\mathscr{S}(4,\mathbb{R})$ with:

$$\{\Sigma_A\xi\} = \{\frac{1}{2}i\gamma_\mu, \Gamma_a\}$$

Concerning group constants we have now:

$$[\Sigma_A \xi, \Sigma_B \xi] \stackrel{\text{dem}}{=} C_{AB}{}^C \Sigma_C \xi \tag{19}$$

with the real $C_{AB}{}^{C}$ that can be computed with the same Tr() technique as in (34). If we define the C_{A} adjoint ten 10x10 real matrices with:

$$(C_A)_B^C \stackrel{\text{def}}{=} C_{AB}^C$$

by using ten Θ^A parameters, we have the exponential relation:

$$e^{\Theta^C \Sigma_C \xi} \left(\Sigma_A \xi \right) e^{-\Theta^D \Sigma_D \xi} \stackrel{\text{dem}}{=} \left(e^{\Theta^E C_E} \right)^B_A \left(\Sigma_B \xi \right)$$

13.1 The $\Sigma_A \xi$ Dirac like equation

If considering that the $\{\Sigma_A\}$ set is more complete than the $\{S_a\}$ set as a basis of 4x4 real symmetric matrices, we can also write:

$$h^{\mu A}(x) \Sigma_A \xi \,\partial_\mu \psi(x) = \frac{mc}{\hbar} \psi(x) \tag{20}$$

But here, since the associated group involves ten Θ^A parameters, we can't define a coordinate Lorentz transformation using η . Instead, we can continue to speak of an internal transformation \mathscr{T} that let the x^{μ} invariant but transforms $h^{\mu A}(x)$ and $\psi^{\alpha}(x)$ such that:

$$\mathcal{T}^{c}(\Theta|x)^{\mu} \stackrel{\text{def}}{=} x^{\mu} = \delta^{\mu}_{\nu} x^{\nu}$$
$$\mathcal{T}^{o}(\Theta|\psi)^{\alpha}(x) \stackrel{\text{def}}{=} (e^{\Theta^{D}\Sigma_{D}\xi})^{\alpha}_{\beta}\psi^{\beta}(x)$$
$$\mathcal{T}^{\bullet\square}(\Theta|h)^{\mu A}(x) \stackrel{\text{def}}{=} (e^{\Theta^{D}C_{D}})^{A}_{B}h^{\mu B}(x) = \delta^{\mu}_{\nu}(e^{\Theta^{D}C_{D}})^{A}_{B}h^{\nu B}(x)$$

14 Connecting to the fifth dimension

We have a remarkable connection between η , the ten $\Sigma_A \xi$ 4x4 real matrices and a similar logic around a five dimensional 5x5 real matrix metric defined with:

$$\tilde{\eta} \stackrel{\text{def}}{=} \left(\begin{array}{ccc} \eta & 0\\ 0 & 1 \end{array} \right) \stackrel{\text{def}}{=} \left(\begin{array}{cccc} 1 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

and a basis $\tilde{A}_{A=1,10}$ of the ten 5x5 real antisymmetric matrices defined with:

$$\tilde{A}_{4+a(=1,2,3,4,5,6)=5,6,7,8,9,10} \stackrel{\text{def}}{=} \left(\begin{array}{cc} A_a & 0\\ 0 & 0 \end{array} \right)$$

It appears that:

$$[\tilde{A}_A \tilde{\eta}, \tilde{A}_B \tilde{\eta}] \stackrel{\text{dem}}{=} C_{AB}{}^C \tilde{A}_C \tilde{\eta}$$

with the same group constants as in (19): nicely strange! Since the number of parameters in a (n=4)x4 symmetric matrix is $\frac{n(n+1)}{2} = 10$ and the number of parameters for a (d=n+1=5)x5 antisymmetric one is $\frac{d(d-1)}{2} = \frac{(n+1)n}{2} = 10$, it is not surprising to find the same number of parameters, but it is more surprising to see that both set of ten 4x4 matrices $\{\Sigma_A\xi\}$ and the upper ten 5x5 $\{\tilde{A}_A\tilde{\eta}\}$ matrices give exactly the same group constants.

14.1 The five dimensional Dirac like equation

If, in the below, the index μ and $\nu=0,1,2,3$ are taken for A=1,2,3,4 and a,b,c=1,2,3,4,5,6 for A=5,6,7,8,9,10, we can check that:

$$\begin{split} & [\tilde{A}_a \tilde{\eta} \tilde{A}_b \tilde{\eta}] \stackrel{\text{dem}}{=} l_{ab}{}^c \tilde{A}_c \tilde{\eta} \\ & [\tilde{A}_a \tilde{\eta}, \tilde{A}_\mu \tilde{\eta}] \stackrel{\text{dem}}{=} (A_a \eta)^{\nu}_{\mu} \tilde{A}_\nu \tilde{\eta} \end{split}$$

We have then, with 5x5 real matrices:

$$e^{\theta^{a}\tilde{A}_{a}\tilde{\eta}}\left(\tilde{A}_{\mu}\tilde{\eta}\right)e^{-\theta^{b}\tilde{A}_{b}\tilde{\eta}} \stackrel{\text{dem}}{=} \left(e^{\theta^{c}A_{c}\eta}\right)^{\nu}_{\mu}\left(\tilde{A}_{\nu}\tilde{\eta}\right)$$

similar to (13), so that we can write an equation similar to the Dirac one but on a $\Psi^{\alpha=0,1,2,3,5}(x)$ tuple:

$$(\eta^{-1})^{\mu}_{\alpha}\tilde{A}_{\alpha}\tilde{\eta}\,\partial_{\mu}\Psi(x) = \frac{mc}{\hbar}\Psi(x) \tag{21}$$

with $\Psi(x)$ transforming in a Lorentz transformation with:

$$\mathscr{L}^{5\bullet}(\theta|\Psi)^{\alpha}(\mathscr{L}^{c}(\theta|x)) \stackrel{\text{\tiny def}}{=} (e^{\theta^{a}\tilde{A}_{a}\tilde{\eta}})^{\alpha}_{\beta}\Psi^{\beta}(x)$$

In this equation there is no exotic complex gamma matrices and the spinor transformation in four dimensions becomes a more natural vector transformation in five dimensions!

15 A word on electromagnetism

Thanks to the Majorana representation of the gamma matrices, we have been able to get rid of complex numbers in our various equations, but traditionally the sticky "i" appears also when handling electromagnetism by writing:

$$i\gamma^{\mu}\{\partial_{\mu}\psi(x) + i\frac{q}{\hbar c}\Phi_{\mu}(x)\psi(x)\} = \frac{mc}{\hbar}\psi(x)$$
(22)

with q being the electromagnetic charge and $\Phi_{\mu}(x)$ being the electromagnetic potential related to the three dimensional Maxwell real $U(t, \vec{x})$ and $\vec{A}(t, \vec{x})$ with:

$$\Phi_{\mu}(x \stackrel{\text{\tiny def}}{=} (ct, \vec{x})) \stackrel{\text{\tiny def}}{=} (U(t, \vec{x}), -\vec{A}(t, \vec{x}))$$

The $\Phi_{\mu}(x)$ would be Lorentz transformed with:

$$\mathscr{L}_{\bullet}(A|\Phi)_{\mu}(\mathscr{L}^{c}(A|x)) \stackrel{\text{def}}{=} (e^{-A\eta})^{\nu}_{\mu} \Phi_{\nu}(x)$$

By using our $\tilde{S}_{\alpha}\xi$ representation, it becomes:

$$h^{\mu\alpha}\,\tilde{S}_{\alpha}\xi\,\{\partial_{\mu}\psi(x) + i\frac{q}{\hbar c}\Phi_{\mu}(x)\psi(x)\} = \frac{mc}{\hbar}\psi(x) \tag{23}$$

which exhibits the fact that "i" appears now only in the electromagnetic coupling to $\Phi_{\mu}(x)$. The complex coupling induces that $\psi(x)$ has to be complex, but if writing:

$$\psi[\mathscr{V}, \mathscr{W}](x) \stackrel{\text{\tiny def}}{=} \mathscr{V}(x) + i\mathscr{W}(x)$$

we see that (23) can be written as two equations without complex numbers on two real coupled fields $\mathscr{V}(x)$ and $\mathscr{W}(x)$:

$$h^{\mu\alpha} \tilde{S}_{\alpha} \xi \left\{ \partial_{\mu} \mathscr{V}(x) - \frac{q}{\hbar c} \Phi_{\mu}(x) \mathscr{W}(x) \right\} = \frac{mc}{\hbar} \mathscr{V}(x)$$
$$h^{\mu\alpha} \tilde{S}_{\alpha} \xi \left\{ \partial_{\mu} \mathscr{W}(x) + \frac{q}{\hbar c} \Phi_{\mu}(x) \mathscr{V}(x) \right\} = \frac{mc}{\hbar} \mathscr{W}(x)$$

or:

$$[h^{\mu\alpha}\,\tilde{S}_{\alpha}\xi\,\partial_{\mu} - \frac{mc}{\hbar}I]\mathscr{V}(x) = [\frac{q}{\hbar c}\,\Phi_{\mu}(x)\,h^{\mu\alpha}\,\tilde{S}_{\alpha}\xi]\mathscr{W}(x) \tag{24}$$

$$[h^{\mu\alpha}\,\tilde{S}_{\alpha}\xi\,\partial_{\mu} - \frac{mc}{\hbar}I]\mathscr{W}(x) = -[\frac{q}{\hbar c}\,\Phi_{\mu}(x)\,h^{\mu\alpha}\,\tilde{S}_{\alpha}\xi]\mathscr{V}(x) \tag{25}$$

We see now that complex numbers appear for electromagnetism in (22) mainly to write in a more compact form two coupled real quantities.

15.1 Charge conjugation transformation \mathscr{C}

With (22), charge conjugation transformation would consist to find a transformation $\mathscr{C}(\psi)$ such that:

$$i\gamma^{\mu}\{\partial_{\mu} - i\frac{q}{\hbar c}\Phi_{\mu}(x)\}\mathscr{C}(\psi)(x) = \frac{mc}{\hbar}\mathscr{C}(\psi)(x)$$
(26)

By using the Dirac (or Chiral) representation of the gamma matrices, we get that:

$$\mathscr{C}(\psi)(x) \stackrel{\text{dem}}{=} i\gamma^2 \psi^*(x) \tag{27}$$

but by working with (24, 25), we see that:

$$\mathscr{C}(\mathscr{V},\mathscr{W})(x) \stackrel{\text{dem}}{=} (\mathscr{W},\mathscr{V})(x)$$
(28)

that is to say the charge conjugation is reduced to just a swapping of the two coupled fields! We find this much more appealing than the much more algebraic (27). (On $\psi(x)$, (28) would be written $\mathscr{C}(\psi)(x) \stackrel{\text{dem}}{=} i\psi^*(x)$).

15.2 \mathscr{C} transformation and our other Dirac like equations

The same conclusion would be reached with our other equations (16), (18), (20) and even with the five dimensioal (21) one. Here too, electromagnetism could be handled by two coupled real fields.

15.3 About Maxwell equations

When writing Maxwell equations in a four dimensional form, it appears equations like:

$$\partial_{\mu}F^{\mu\nu}(x) = J^{\nu}(x)$$

where $F^{\mu\nu}(x)$ is antisymmetric. This equation is Lorentz invariant with:

$$\mathscr{L}^{\bullet}(\theta|J)^{\mu}(\mathscr{L}^{c}(\theta|x)) \stackrel{\text{def}}{=} (e^{\theta^{a}A_{a}\eta})^{\mu}_{\nu}J^{\nu}(x)$$
$$\mathscr{L}^{\bullet\bullet}(\theta|F)^{\mu\nu}(\mathscr{L}^{c}(\theta|x)) \stackrel{\text{def}}{=} (e^{\theta^{a}A_{a}\eta})^{\mu}_{\alpha}(e^{\theta^{b}A_{b}\eta})^{\nu}_{\beta}F^{\alpha\beta}(x)$$
(29)

Since $F^{\mu\nu}$ is antisymmetric, we can introduce the six $F^a(x)$ fields such that:

$$F^{\mu\nu}(x) \stackrel{\text{\tiny def}}{=} (A_a)^{\mu}_{\nu} F^a(x)$$

With this, the upper equation becomes:

$$(A_a)^{\mu}_{\nu}\partial_{\mu}F^a(x) = J^{\nu}(x) \tag{30}$$

which reminds the (18) equation.

Since (see Appendix B with $S = \eta$) we have:

$$e^{\theta^c A_c \eta} A_a{}^t (e^{\theta^d A_d \eta}) \stackrel{\text{dem}}{=} (e^{\theta^e L_e})^b_a A_b$$

We can show that (29) induces:

$$\mathscr{L}^{\Box}(\theta|F)^{a}(\mathscr{L}^{c}(\theta|x)) \stackrel{\text{dem}}{=} (e^{\theta^{c}L_{c}})^{a}_{b}F^{b}(x)$$

which let (30) Lorentz invariant.

16 Toward six or ten dimensional "adjoint" spacetime?

In the case of (14) and (21), h(x) is a constant and a square (4x4) real matrix, but in case of (16), (18) and (20) it is not a square matrix (4x6 or 4x10) and a priori not a constant field. Being not square is not handy, but we can recover that by just going to a space-time with six or ten dimensions! Indeed, if instead of $(x^{\mu=0,1,2,3})$ coordinates we go toward $(X^{a=1\rightarrow 6})$ or $(X^{A=1\rightarrow 10})$, then (16) or (20) become:

$$h^{ab}(X) S_b \xi \partial_a \psi(X) = \frac{mc}{\hbar} \psi(X)$$
$$h^{AB}(X) \Sigma_B \xi \partial_A \psi(X) = \frac{mc}{\hbar} \psi(X)$$

They will be invariant with the transformations:

$$\mathcal{T}^{c}(\theta|X)^{a} \stackrel{\text{def}}{=} (e^{\theta^{d}L_{d}})^{a}_{b}X^{b}$$
$$\mathcal{T}^{o}(\theta|\psi)^{\alpha}(\mathcal{T}^{c}(\theta|X)) \stackrel{\text{def}}{=} (e^{\theta^{a}S_{a}\xi})^{\alpha}_{\beta}\psi^{\beta}(X)$$
$$\mathcal{T}^{\Box\Box}(\theta|h)^{ab}(\mathcal{T}^{c}(\theta|X)) \stackrel{\text{def}}{=} (e^{\theta^{c}L_{c}})^{a}_{e}(e^{\theta^{d}L_{d}})^{b}_{f}h^{ef}(X)$$

or:

$$\mathcal{T}^{c}(\Theta|X)^{A} \stackrel{\text{def}}{=} (e^{\Theta^{D}C_{D}})^{A}_{B}X^{B}$$
$$\mathcal{T}^{\circ}(\Theta|\psi)^{\alpha}(\mathcal{T}^{c}(\Theta|X)) \stackrel{\text{def}}{=} (e^{\Theta^{A}\Sigma_{A}\xi})^{\alpha}_{\beta}\psi^{\beta}(X)$$
$$\mathcal{T}^{\Box\Box}(\Theta|h)^{AB}(\mathcal{T}^{c}(\Theta|X)) \stackrel{\text{def}}{=} (e^{\Theta^{C}C_{C}})^{A}_{E}(e^{\Theta^{D}C_{D}})^{B}_{F}h^{EF}(X)$$

We can even show that we can have constant h(X) fields. Indeed, if taking:

$$h^{ab}(X) \stackrel{\text{\tiny def}}{=} G^a_b$$

with the G 6x6 real matrix defined by:

$$G \stackrel{\text{def}}{=} G_6 \stackrel{\text{def}}{=} \begin{pmatrix} I_{3x3} & 0\\ 0 & -I_{3x3} \end{pmatrix} \quad I_{3x3} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix}$$

we have:

$$\mathscr{T}^{\Box\Box}(\theta|h)^{ab}(\mathscr{T}^{c}(\theta|X)) \stackrel{\text{dem}}{=} h^{ab}$$

and the same for $h^{AB}(X)$ with the G_{10} 10x10 real matrix defined by:

$$h^{AB}(X) \stackrel{\text{\tiny def}}{=} (G_{10})^A_B \quad G_{10} \stackrel{\text{\tiny def}}{=} \left(\begin{array}{cc} \eta & 0\\ 0 & G_6 \end{array} \right)$$

17 Checked numerically by computer

All the formulas used in this article had been checked numerically by computer. On the author GitHub gbarrand, the repository "papers" contains the open source "SMAD" C++ unitary test program that verifies the formulas found here.

18 Conclusions

Thanks to E.Majorana, it is interesting to see that the four dimensional vector and spinor Lorentz transformations show a similar "AS/SA" real matrix pattern (S=Symmetric, A=Antisymmetric), and that the "SA" real pattern for the spinor transformation is connected to an "AS" real pattern in fifth dimension. With this in head, we see now the Dirac spinor Lorentz transformation differently; instead of being related to some exotic mathematical trick through complex Dirac matrices introduced through (5), we see it now deeply related to a real antisymmetric ξ matrix which is the pending of the real symmetric η matrix for the four vector transformation. By pushing to the fifth dimension, we can even see it as a five vector transformation. Moreover, following the logic around this ξ matrix, we see that the traditional Dirac equation looks rather incomplete and could be extended in a more natural way to something as (16) or (20) that we write again here:

$$h^{\mu a}(x) S_a \xi \partial_\mu \psi(x) = \frac{mc}{\hbar} \psi(x)$$
$$h^{\mu A}(x) \Sigma_A \xi \partial_\mu \psi(x) = \frac{mc}{\hbar} \psi(x)$$

with the $\psi(x)$ tuple not needed to be complex at this point, and the coworking h(x) real fields coming naturally from a group point of view. We saw also that electromagnetism could be handled without complex numbers in all our Dirac like equations by just introducing two coupled real fields. Finally, we showed that going toward six or ten dimensional spacetime would be more natural from a group point of view.

Appendix A γ^{μ} in Majorana representation

The $\gamma^{\mu=0,1,2,3}$ in the Majorana representation are presented in general as (see [1] p.694):

$$\gamma^{0} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & \sigma_{2} \\ \sigma_{2} & 0 \end{pmatrix} \quad \gamma^{1} \stackrel{\text{def}}{=} \begin{pmatrix} i\sigma_{3} & 0 \\ 0 & i\sigma_{3} \end{pmatrix}$$
$$\gamma^{2} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & -\sigma_{2} \\ \sigma_{2} & 0 \end{pmatrix} \quad \gamma^{3} \stackrel{\text{def}}{=} \begin{pmatrix} -i\sigma_{1} & 0 \\ 0 & -i\sigma_{1} \end{pmatrix}$$

with the standard Pauli 2x2 complex matrices being defined as:

$$\sigma_1 \stackrel{\text{def}}{=} \left(\begin{array}{c} 0 & 1 \\ 1 & 0 \end{array} \right) \qquad \sigma_2 \stackrel{\text{def}}{=} \left(\begin{array}{c} 0 & -i \\ i & 0 \end{array} \right) \qquad \sigma_3 \stackrel{\text{def}}{=} \left(\begin{array}{c} 1 & 0 \\ 0 & -1 \end{array} \right)$$

With less complex numbers around, we can write them directly as:

$$\gamma^{0} \stackrel{\text{def}}{=} i \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \gamma^{1} \stackrel{\text{def}}{=} i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$
$$\gamma^{2} \stackrel{\text{def}}{=} i \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \gamma^{3} \stackrel{\text{def}}{=} i \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Appendix B $\gamma_{\mu}, \Gamma[A]$ matrices

Let us have γ_{μ} and a S real symmetric square matrices such that:

$$\{\gamma_{\mu}, \gamma_{\nu}\} \stackrel{\text{\tiny def}}{=} 2S^{\mu}_{\nu}I$$

I being the identity matrix. Note that we do not specify the dimensions of them! The γ_{μ} can be real or complex. With a little bit of algebra, we can show that it induces:

$$[[\gamma_{\mu}, \gamma_{\nu}], \gamma_{\alpha}] \stackrel{\text{dem}}{=} 4S^{\alpha}_{\nu}\gamma_{\mu} - 4S^{\alpha}_{\mu}\gamma_{\nu}$$
(31)

If A being a real antisymmetric square matrix of same dimension as S, let us define $\Gamma[A]$:

$$\Gamma[A] \stackrel{\text{\tiny def}}{=} \frac{1}{8} [\gamma_{\mu}, \gamma_{\nu}] (A)^{\mu}_{\nu}$$

The relation (31) induces:

$$[\Gamma[A], \gamma_{\mu}] \stackrel{\text{\tiny dem}}{=} (AS)^{\nu}_{\mu} \gamma_{\nu}$$

It is a general property (see [1] p.70) that for square matrices M, N_a of same dimension:

$$[M, N_a] \stackrel{\text{def}}{=} C^b_a N_b \quad \Rightarrow \quad e^M N_a e^{-M} \stackrel{\text{dem}}{=} (e^C)^b_a N_b \tag{32}$$

Then we have the important property:

$$e^{\Gamma[A]}\gamma_{\mu}e^{-\Gamma[A]} \stackrel{\text{dem}}{=} (e^{AS})^{\nu}_{\mu}\gamma_{\nu}$$

B.1 γ^{μ} "up" matrices

If defining γ^{μ} with:

$$\gamma^{\mu} \stackrel{\text{def}}{=} = (S^{-1})^{\mu}_{\nu} \gamma_{\nu}$$

we have:

$$\{\gamma^{\mu}, \gamma^{\nu}\} \stackrel{\text{dem}}{=} 2(S^{-1})^{\mu}_{\nu}I$$
$$\Gamma[A] \stackrel{\text{dem}}{=} \frac{1}{8}[\gamma^{\mu}, \gamma^{\nu}](SAS)^{\mu}_{\nu}$$
$$[\Gamma[A], \gamma^{\mu}] \stackrel{\text{dem}}{=} (-AS)^{\mu}_{\nu}\gamma^{\nu}$$

It is a general property (see [1] p.70) that for square matrices M, N^a of same dimension:

$$[M, N^a] \stackrel{\text{\tiny def}}{=} C^a_b N^b \quad \Rightarrow \quad e^M N^a e^{-M} \stackrel{\text{\tiny dem}}{=} (e^C)^a_b N^b$$

Then we have the important property:

$$e^{\Gamma[A]}\gamma^{\mu}e^{-\Gamma[A]} \stackrel{\text{\tiny dem}}{=} (e^{-AS})^{\mu}_{\nu}\gamma^{\nu}$$

B.2 The A_a basis

Let us define the antisymmetric $A_{a=1,2,3,4,5,6}$ six matrices with:

These matrices form a basis for any 4x4 real antisymmetric matrix. We can show that:

$$Tr(A_aA_b) \stackrel{\text{dem}}{=} -2\delta_{ab}$$

B.3 A_aS , adjoint matrices

It appears that:

$$A_a S A_b - A_b S A_a$$

is antisymmetric, and then we can develop it on the A_a basis and write:

$$A_a S A_b - A_b S A_a \stackrel{\text{dem}}{=} C_{ab}{}^c A_c \Rightarrow \quad [A_a S, A_b S] \stackrel{\text{dem}}{=} C_{ab}{}^c A_c S \tag{33}$$

The $C_{ab}{}^c$ can be computed by using the matrix trace Tr() with:

$$Tr[A_d(A_aSA_b - A_bSA_a)] = C_{ab}{}^c Tr(A_dA_c) = C_{ab}{}^c (-2\delta_{dc}) \stackrel{\text{dem}}{=} -2C_{ab}{}^d$$
(34)

By defining the C_a "adjoint" matrices:

$$(C_a)_c^b \stackrel{\text{\tiny def}}{=} C_{ac}{}^b$$

we have the commutators:

$$[C_a, C_b] \stackrel{\text{dem}}{=} C_{ab}{}^c C_c$$

and by exploiting (32) we have:

$$e^{\theta^{c}A_{c}S} A_{a}S e^{-\theta^{d}A_{d}S} \stackrel{\text{dem}}{=} (e^{\theta^{e}C_{e}})^{b}_{a} A_{b}S$$
$$e^{\theta^{c}C_{c}} C_{a} e^{-\theta^{d}C_{d}} \stackrel{\text{dem}}{=} (e^{\theta^{e}C_{e}})^{b}_{a} C_{b}$$

If S is invertible, from the first upper relation we can deduce that we have also:

$$e^{\theta^c A_c S} A_a {}^t (e^{\theta^d A_d S}) \stackrel{\text{dem}}{=} (e^{\theta^e C_e})^b_a A_b$$

B.4 Γ_a matrices

By using the upper A_a real matrices, we define:

$$\Gamma_a \stackrel{\text{def}}{=} \frac{1}{8} [\gamma_\mu, \gamma_\nu] (A_a)^{\mu}_{\nu}$$

We can demonstrate that the commutators (33) induce:

$$[\Gamma_a, \Gamma_b] \stackrel{\text{dem}}{=} C_{ab}{}^c \Gamma_c$$

with the same real group constants $C_{ab}{}^{c}$. By exploiting (32) we have also:

$$e^{\theta^c \Gamma_c} \Gamma_a e^{-\theta^d \Gamma_d} \stackrel{\text{dem}}{=} (e^{\theta^e C_e})^b_a \Gamma_b$$

B.5 The Baker–Campbell–Hausdorff development

If X and Y are matrices of same dimension, we have:

$$e^X e^Y = e^Z$$

with:

$$Z[X,Y] \stackrel{\text{dem}}{=} X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \dots$$

B.6 The \oplus group operation

We can then define the \oplus group operation over some θ_1^a and θ_1^a parameters such that:

$$(\theta_1 \oplus \theta_2)^a \stackrel{\text{dem}}{=} \theta_1^a + \theta_2^a + \frac{1}{2}C_{bc}{}^a\theta_1^b\theta_2^c + \frac{1}{12}C_{ed}{}^aC_{bc}{}^d\theta_1^e\theta_1^b\theta_2^c - \frac{1}{12}C_{ed}{}^aC_{bc}{}^d\theta_2^e\theta_1^b\theta_2^c + \dots$$

following a Baker–Campbell–Hausdorff development. It permits to have:

$$e^{\theta_1^a A_a S} e^{\theta_2^b A_b S} \stackrel{\text{dem}}{=} e^{(\theta_1 \oplus \theta_2)^a A_a S}$$
$$e^{\theta_1^a \Gamma_a} e^{\theta_2^b \Gamma_b} \stackrel{\text{dem}}{=} e^{(\theta_1 \oplus \theta_2)^a \Gamma_a}$$

which enforce that e^{AS} and $e^{\Gamma[A]}$ are two representations of the same group.

Appendix C $l_{ab}{}^c$ and L_a matrices

With the real $l_{ab}{}^c$ introduced through:

$$[A_a\eta, A_b\eta] \stackrel{\text{dem}}{=} l_{ab}{}^c A_c\eta$$

and the L_a adjoint 6x6 real matrices defined with:

$$(L_a)_c^b \stackrel{\text{def}}{=} l_{ac}^{\ b}$$

we have:

$$L_{j=1,2,3} \stackrel{\text{dem}}{=} \begin{pmatrix} -\epsilon_j & 0\\ 0 & -\epsilon_j \end{pmatrix} \quad L_{a=4,5,6=j+3} \stackrel{\text{dem}}{=} \begin{pmatrix} 0 & \epsilon_j\\ -\epsilon_j & 0 \end{pmatrix}$$

with:

$$\epsilon_1 \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad \epsilon_2 \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \epsilon_3 \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The L_a have all the properties shown in the Appendix B with $S = \eta$, In particular we have:

$$e^{\theta^{c}A_{c}\eta} A_{a}\eta e^{-\theta^{d}A_{d}\eta} \stackrel{\text{dem}}{=} (e^{\theta^{e}L_{e}})_{a}^{b} A_{b}\eta$$
$$e^{\theta^{c}\Gamma_{c}} \Gamma_{a} e^{-\theta^{d}\Gamma_{d}} \stackrel{\text{dem}}{=} (e^{\theta^{e}L_{e}})_{a}^{b} \Gamma_{b}$$
$$e^{\theta^{c}L_{c}} L_{a} e^{-\theta^{d}L_{d}} \stackrel{\text{dem}}{=} (e^{\theta^{e}L_{e}})_{a}^{b} L_{b}$$

Appendix D \mathscr{R} transformation, Vierbein

D.1 General frame transformation \mathscr{R}

One feature of general relativity is the invariance of formulas according to a general frame transformation. The idea behind this being that laws of physics should look the same in any reference frame. With the same kind of notation used in the first paragraph (1), a general coordinate transformation on the tuple $x^{\mu=0,1,2,3}$ can be written:

$$\mathscr{R}^{c}(r|x)^{\mu} \stackrel{\text{\tiny def}}{=} r^{\mu}(x)$$

the $r^{\mu}(x)$ being four "well behaved" functions (of the x^{μ}) that are seen as the parameters of the transformation \mathscr{R} (similar as an antisymmetric matrix A is seen as the parameter of a Lorentz transformation). If we introduce the notation:

$$R[r]^{\mu}_{\nu}(x) \stackrel{\text{\tiny def}}{=} \partial_{\nu}\{r^{\mu}\}(x) \qquad \tilde{R}[r]^{\mu}_{\nu}(x) \stackrel{\text{\tiny def}}{=} \partial_{\nu}\{(r^{-1})^{\mu}\}(r(x))$$

we have:

$$\tilde{R}^{\mu}_{\alpha}(x)R^{\alpha}_{\nu}(x) \stackrel{\text{dem}}{=} \partial_{\nu}\{(r^{-1}\circ r)^{\mu}(x)\}(x) \stackrel{\text{dem}}{=} \partial_{\nu}\{x^{\mu}\} \stackrel{\text{dem}}{=} \delta^{\mu}_{\nu}$$
$$R^{\mu}_{\alpha}(x)\tilde{R}^{\alpha}_{\nu}(x) \stackrel{\text{dem}}{=} \partial_{\nu}\{(r\circ r^{-1})^{\mu}(x)\}(r(x)) \stackrel{\text{dem}}{=} \delta^{\mu}_{\nu}$$

With this, a general frame transformation for a $\phi(x)$, a $V^{\mu}(x)$, a $T^{\mu\nu}(x)$, a $T_{\mu\nu}(x)$, etc, reads:

$$\mathscr{R}^{s}(r|\phi)(\mathscr{R}^{c}(r|x)) \stackrel{\text{def}}{=} \phi(x)$$

$$\mathscr{R}^{\bullet}(r|V)^{\mu}(\mathscr{R}^{c}(r|x)) \stackrel{\text{def}}{=} R^{\mu}_{\nu}(x)V^{\mu}(x)$$

$$\mathscr{R}^{\bullet\bullet}(r|T)^{\mu\nu}(\mathscr{R}^{c}(r|x)) \stackrel{\text{def}}{=} R^{\mu}_{\alpha}(x)R^{\nu}_{\beta}(x)T^{\alpha\beta}(x)$$

$$\mathscr{R}_{\bullet\bullet}(r|T)_{\mu\nu}(\mathscr{R}^{c}(r|x)) \stackrel{\text{def}}{=} \tilde{R}^{\alpha}_{\mu}(x)\tilde{R}^{\beta}_{\nu}(x)T_{\alpha\beta}(x)$$
(35)

For all the upper definitions, we have the important composition group property:

$$\mathscr{R}(r_1|\mathscr{R}(r_2|whatever)) \stackrel{\text{dem}}{=} \mathscr{R}(r_1 \circ r_2|whatever)$$

We can note that for these, a Lorentz transformation \mathscr{L} is a particular case of a \mathscr{R} if we take:

$$r^{\mu}(x) \stackrel{\text{\tiny def}}{=} (e^{A\eta})^{\mu}_{\nu} x^{\nu}$$

If, by using a $g_{\mu\nu}(x)$, we define the length of of a $V^{\mu}(x)$ with:

$$l[g](V)(x) \stackrel{\text{\tiny def}}{=} g_{\mu\nu}(x)V^{\mu}(x)V^{\nu}(x)$$

with the upper definitions (35), it is easy to show that it is invariant by a general frame transformation:

$$l[\mathscr{R}_{\bullet\bullet}(r|g)](\mathscr{R}^{\bullet}(r|V))(\mathscr{R}^{c}(r|x)) \stackrel{\text{def}}{=} l[g](V)(x)$$

D.2 Spinor transformation? Vierbein

Concerning a spinor transformation \mathscr{R}° , we have a slight problem since we have no obvious extension of the Lorentz transformation \mathscr{L}° (4) for them. To handle the Dirac equation within general relativity, people do a (not so appealing) compound construction by introducing "vierbein" fields $e^{\alpha}_{\mu}(x)$ bearing two indices of different kind concerning transformations. The vierbein fields are considered to be more fundamental than the metric $g_{\mu\nu}(x)$, and in fact define the metric with:

$$g_{\mu\nu}(x) \stackrel{\text{def}}{=} e^{\alpha}_{\mu}(x)e^{\beta}_{\nu}(x)\eta_{\alpha\beta} \tag{36}$$

Are introduced also co-vierbein fields $\tilde{e}^{\mu}_{\alpha}(x)$ such that:

$$e^{\alpha}_{\mu}(x)\tilde{e}^{\mu}_{\beta}(x) \stackrel{\text{\tiny def}}{=} \delta^{\alpha}_{\beta} \qquad \tilde{e}^{\mu}_{\alpha}(x)e^{\alpha}_{\nu}(x) \stackrel{\text{\tiny def}}{=} \delta^{\mu}_{\nu}$$

D.3 Local (or internal) Lorentz transformation \mathscr{L}_{I}

Concerning transformations for $e^{\alpha}_{\mu}(x)$, the \mathscr{R} one is applied for the μ down index, but an "internal" or "local" Lorentz one \mathscr{L}_{I} is applied for the α upper one. The \mathscr{L}_{I} is defined as:

$$\mathscr{L}_{I}^{c}(A|x)^{\mu} \stackrel{\mathrm{def}}{=} x^{\mu}$$

(then no effect on x^{μ}), and on a $\psi(x)$:

$$\mathscr{L}_{I}^{\circ}(A|\psi)^{\alpha}(x) \stackrel{\text{\tiny def}}{=} (e^{\Gamma[A]})^{\alpha}_{\beta}\psi^{\beta}(x)$$

We note then the important difference with a \mathscr{L} ; the x^{μ} are not changed with a \mathscr{L}_{I} transformation.

D.4 Dirac equation with vierbein

The Dirac equation on "curved spacetime" is now written:

$$\tilde{e}^{\mu}_{\alpha}(x)i\gamma^{\alpha}\partial_{\mu}\psi(x) = \frac{mc}{\hbar}\psi(x)$$
(37)

Someone can check that this equation is invariant by both the \mathscr{R} and \mathscr{L}_I transformations with: $\mathscr{O}(\mathcal{O}(\mathcal{O}(\mathbb{C}))) \stackrel{\text{def}}{\longrightarrow} (\mathcal{O}(\mathbb{C}))$

$$\mathscr{R}^{s}(r|\psi)^{\alpha}(\mathscr{R}^{c}(r|x)) \stackrel{\text{def}}{=} \psi^{\alpha}(x)$$
$$\mathscr{L}^{\circ}_{I}(A|\psi)^{\alpha}(x) \stackrel{\text{def}}{=} (e^{\Gamma[A]})^{\alpha}_{\beta}\psi^{\beta}(x)$$

$$\mathscr{R}_{\bullet}(r|e^{\alpha})_{\mu}(\mathscr{R}^{c}(r|x)) \stackrel{\text{def}}{=} \overset{R}{R}^{\nu}_{\mu}(x)e^{\alpha}_{\nu}(x)$$

$$\begin{aligned} \mathscr{L}_{I}^{\circ}(A|e_{\mu})^{\alpha}(x) &\stackrel{\text{def}}{=} (e^{A\eta})^{\alpha}_{\beta}e^{\beta}_{\mu}(x) \\ \mathscr{R}^{s}(r|e_{\mu})^{\alpha}(\mathscr{R}^{c}(r|x)) &\stackrel{\text{def}}{=} e^{\alpha}_{\mu}(x) \\ \mathscr{L}_{I}^{s}(A|e^{\alpha})_{\mu}(x) &\stackrel{\text{def}}{=} e^{\alpha}_{\mu}(x) \end{aligned}$$
$$\begin{aligned} \mathscr{R}^{\bullet}(r|\tilde{e}_{\alpha})^{\mu}(\mathscr{R}^{c}(r|x)) &\stackrel{\text{def}}{=} R^{\mu}_{\nu}(x)\tilde{e}^{\nu}_{\alpha}(x) \\ \mathscr{L}_{I\circ}(A|\tilde{e}^{\mu})_{\alpha}(x) &\stackrel{\text{def}}{=} (e^{-A\eta})^{\alpha}_{\alpha}\tilde{e}^{\mu}_{\beta}(x) \\ \mathscr{R}^{s}(r|\tilde{e}^{\mu})_{\alpha}(\mathscr{R}^{c}(r|x)) &\stackrel{\text{def}}{=} e^{\mu}_{\alpha}(x) \\ \mathscr{L}_{Is}(A|\tilde{e}_{\alpha})^{\mu}(x) &\stackrel{\text{def}}{=} \tilde{e}^{\mu}_{\alpha}(x) \end{aligned}$$

To play with the upper transformations over the vierbein and co-vierbein fields, we can show that with the metric defined by (36) we have:

$$\mathscr{R}_{\bullet\bullet}(r|g)_{\mu\nu}(x) \stackrel{\text{dem}}{=} \mathscr{R}_{\bullet}(r|e^{\alpha})_{\mu}(x) \mathscr{R}_{\bullet}(r|e^{\beta})_{\nu}(x) \eta_{\alpha\beta}$$

and:

$$\mathscr{L}_{Is}(A|g)_{\mu\nu}(x) \stackrel{\text{\tiny def}}{=} g_{\mu\nu}(x) \stackrel{\text{\tiny dem}}{=} \mathscr{L}_{I}^{\circ}(A|e_{\mu})^{\alpha}(x) \mathscr{L}_{I}^{\circ}(A|e_{\nu})^{\beta}(x) \eta_{\alpha\beta}$$

The compound construction (37) is probably related to the fact that, put all together, it makes no sense to attempt to mix the Dirac equation, related to the "quantum world", with maths related to the macroscopic world modeled with general relativity, and this without having unified in first place the ideas related to these two worlds. (A "quantum gravity" theory will probably be based on only one single "crystal clear" transformation).

By using the Majorana representation, the equation (37) can be written:

$$h^{\mu\alpha}(x)\,\tilde{S}_{\alpha}\xi\,\partial_{\mu}\psi(x) = \frac{mc}{\hbar}\psi(x)$$
$$h^{\mu\alpha}(x) \stackrel{\text{def}}{=} \tilde{e}^{\mu}_{\beta}(x)(\eta^{-1})^{\beta}_{\alpha}$$

which is similar to (15) but with the $h^{\mu\alpha}(x)$ fields being now not constant and transforming in a mixed way for each index (with \mathscr{R} for the first one and \mathscr{L}_I for the second one). We can reach, here too, the conclusion that this equation is not so natural from a group point of view and something as (16) or (20) would be better.

References

 C.Itzykson, J.B.Zuber *Quantum Field Theory* (McGraw-Hill International Book Company, 1980)